第十三届全国大学生数学竞赛初赛补赛
（非数学类）试题及参考解答

【说明】：这套试卷是因为疫情影响部分赛区延迟比较后的统一竞赛试卷

一、填空题(30 分，每小题 6 分)

1、设 $x_0=1, x_n=\ln\left(1+x_{n-1}\right)(n\geq 1)$，则 $\lim\limits_{n\to+\infty} nx_n=$________.

【参考解答】：由题设可知 $x_n\geq 0$，且由 $\ln\left(1+x\right)<x$，得

$$x_{n+1}-x_n=\ln\left(1+x_n\right)-x_n\leq 0$$

即数列 $\left\{x_n\right\}$ 单调递减. 由单调有界原理可知数列 $\left\{x_n\right\}$ 收敛. 令 $\lim\limits_{n\to\infty} x_n=A$，对递推式两端取极限，得 $A=\ln(1+A)$，故 $A=0$，即 $\lim\limits_{n\to\infty} x_n=0$. 改写极限式并由 Stolz 定理得

$$\lim_{n\to\infty} nx_n=\lim_{n\to\infty}\frac{n}{\frac{1}{x_n}}=\lim_{n\to\infty}\frac{(n+1)-n}{\frac{1}{x_{n+1}}-\frac{1}{x_n}}$$

$$=\lim_{n\to\infty}\frac{x_nx_{n+1}}{x_n-x_{n+1}}=\lim_{n\to\infty}\frac{x_n\ln\left(1+x_n\right)}{x_n-\ln\left(1+x_n\right)}$$

由于 $\lim\limits_{n\to\infty} x_n=0$，故 $\ln\left(1+x_n\right)\sim x_n\left(n\to\infty\right)$. 又

$$\ln\left(1+x\right)=x-\frac{x}{2}+o\left(x^2\right)$$

代入最后计算得到的极限式，得 $\lim\limits_{n\to\infty} nx_n=\lim\limits_{n\to\infty}\dfrac{x_n^2}{\frac{1}{2}x_n^2}=2$.

2、积分 $I=\displaystyle\int_0^{\frac{\pi}{2}}\frac{\cos x}{1+\tan x}\mathrm{d}x=$________.

【参考解答】：【思路一】由万能公式 $\cos x=\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \tan x=\dfrac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$. 令 $t=\arctan\dfrac{x}{2}$，得 $\mathrm{d}x=\dfrac{2}{1+t^2}\mathrm{d}t$，代入得

$$I=-\int_0^1\frac{2\left(t^2-1\right)^2}{\left(t^2+1\right)^2\left(t^2-2t-1\right)}\mathrm{d}t$$

分解部分分式，得

$$\frac{2\left(t^2-1\right)^2}{\left(t^2+1\right)^2\left(t^2-2t-1\right)}=\frac{2(t-1)}{\left(t^2+1\right)^2}+\frac{1}{t^2+1}+\frac{1}{t^2-2t-1}$$

分成三个部分分别积分，得

$$\int \frac{1}{t^2+1}\mathrm{d}t = \arctan t + C$$

$$\int \frac{1}{t^2-2t-1}\mathrm{d}t = \int \frac{\mathrm{d}(t-1)}{(t-1)^2-2}(t-1=u) = \int \frac{\mathrm{d}u}{u^2-2}$$

$$= \frac{1}{2\sqrt{2}}\int\left(\frac{1}{u-\sqrt{2}} - \frac{1}{u+\sqrt{2}}\right)\mathrm{d}u = \frac{1}{2\sqrt{2}}\ln\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\ln\left|\frac{t-1-\sqrt{2}}{t-1+\sqrt{2}}\right| + C$$

故得 $\int_0^1 \frac{1}{t^2+1}\mathrm{d}t = \arctan 1 = \frac{\pi}{4}$，

$$\int_0^1 \frac{1}{t^2-2t-1}\mathrm{d}t = \left[\frac{1}{2\sqrt{2}}\ln\left|\frac{t-1-\sqrt{2}}{t-1+\sqrt{2}}\right|\right]_0^1 = -\frac{1}{2\sqrt{2}}\ln\frac{1+\sqrt{2}}{\sqrt{2}-1}$$

对于积分 $\int \frac{2(t-1)}{(t^2+1)^2}\mathrm{d}t$，拆分为两部分，分别为

$$\int \frac{2(t-1)}{(t^2+1)^2}\mathrm{d}t = \int \frac{2t}{(t^2+1)^2}\mathrm{d}t - \int \frac{2}{(t^2+1)^2}\mathrm{d}t$$

其中 $\int \frac{2t}{(t^2+1)^2}\mathrm{d}t = \int \frac{\mathrm{d}(1+t^2)}{(t^2+1)^2} = -\frac{1}{1+t^2} + C$，故

$$\int_0^1 \frac{2t}{(t^2+1)^2}\mathrm{d}t = \left[-\frac{1}{1+t^2}\right]_0^1 = \frac{1}{2}$$

对于第二个积分，由分部积分法，得

$$\int \frac{2}{(t^2+1)^2}\mathrm{d}t = -\int \frac{1}{t}\mathrm{d}\left(\frac{1}{t^2+1}\right) = -\frac{1}{t}\cdot\frac{1}{t^2+1} - \int \frac{1}{t^2(t^2+1)}\mathrm{d}t$$

$$= -\frac{1}{t(t^2+1)} - \int\left(\frac{1}{t^2} - \frac{1}{t^2+1}\right)\mathrm{d}t = -\frac{1}{t(t^2+1)} + \frac{1}{t} + \arctan t + C$$

代入上下限，得

$$\int_0^1 \frac{2}{(t^2+1)^2}\mathrm{d}t = \left[-\frac{1}{t(t^2+1)} + \frac{1}{t} + \arctan t\right]_0^1$$

$$= -\frac{1}{2} + 1 + \frac{\pi}{4} - \lim_{t\to 0^+}\left[-\frac{1}{t(t^2+1)} + \frac{1}{t} + \arctan t\right]$$

$$= \frac{1}{2} + \frac{\pi}{4} - \lim_{t\to 0^+}\frac{t}{t^2+1} = \frac{1}{2} + \frac{\pi}{4}$$

代入最初需要计算的积分，得

$$I=-\left(\frac{\pi}{4}-\frac{1}{2\sqrt{2}}\ln\frac{1+\sqrt{2}}{\sqrt{2}-1}+\frac{1}{2}-\frac{1}{2}-\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}\ln\frac{1+\sqrt{2}}{\sqrt{2}-1}$$

【思路二】 由 $\tan x=\dfrac{\sin x}{\cos x}$，代入并令 $x=\dfrac{\pi}{2}-t$，得

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\frac{\cos x}{1+\tan x}\mathrm{d}x&=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\sin x+\cos x}\mathrm{d}x\\&=\int_0^{\frac{\pi}{2}}\frac{\sin^2 t}{\sin t+\cos t}\mathrm{d}t=\int_0^{\frac{\pi}{2}}\frac{\sin^2 x}{\sin x+\cos x}\mathrm{d}x\end{aligned}$$

把中间两项相加，得

$$\begin{aligned}I&=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x+\sin^2 x}{\sin x+\cos x}\mathrm{d}x=\int_0^{\frac{\pi}{2}}\frac{1}{\sin x+\cos x}\mathrm{d}x\\&=\frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\left(x+\frac{\pi}{4}\right)}{\sin\left(x+\frac{\pi}{4}\right)}\left(u=x+\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{\mathrm{d}u}{\sin u}\\&=\frac{1}{2\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{-\mathrm{d}\cos u}{1-\cos^2 u}(\cos u=t)\ =\frac{1}{2\sqrt{2}}\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}}\frac{\mathrm{d}t}{1-t^2}\\&=\frac{1}{4\sqrt{2}}\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}}\left(\frac{1}{t+1}-\frac{1}{t-1}\right)\mathrm{d}t=\frac{1}{4\sqrt{2}}\left[\ln\left|\frac{t+1}{t-1}\right|\right]_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}}\\&=\frac{1}{4\sqrt{2}}\left(\ln\frac{\sqrt{2}+2}{2-\sqrt{2}}-\ln\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)=\frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2}+2}{2-\sqrt{2}}\end{aligned}$$

3、已知直线 $L:\begin{cases}2x-4y+z=0\\3x-y-2z=9\end{cases}$ 和平面 $\pi:4x-y+z=1$，则直线 L 在平面 π 上的投影直线方程为________.

【参考解答】: 所求投影直线方程即为过直线 L 且与平面 π 垂直的平面与平面 π 的交线. 过直线 L 的平面束方程为

$$\begin{aligned}&2x-4y+z+\lambda(3x-y-2z-9)\\&=(2+3\lambda)x-(4+\lambda)y+(1-2\lambda)z-9\lambda=0\end{aligned}$$

求与平面 π 垂直的平面束中的方程，则两平面的法向量垂直，即

$$(2+3\lambda,-(4+\lambda),1-2\lambda)\cdot(4,-1,1)=0$$

解得 $\lambda=-\dfrac{13}{11}$. 代入平面束方程，得

$$-\frac{17x}{11}-\frac{31y}{11}+\frac{37z}{11}+9\cdot\frac{13}{11}=0$$

即 $17x+31y-37z-117=0$，故投影直线方程为

$$\begin{cases}17x+31y-37z-117=0\\ 4x-y+z-1=0\end{cases}$$

.

4、$\sum\limits_{n=1}^{+\infty}\arctan\dfrac{2}{4n^2+4n+1}=$________.

【参考解答】：由正切函数恒等变换关系

$$\arctan\frac{x-y}{1+xy}=\arctan x-\arctan y$$

改写$\dfrac{2}{4n^2+4n+1}=\dfrac{(2n+2)-2n}{1+2n(2n+2)}$，得

$$\sum_{n=1}^{+\infty}\arctan\frac{2}{4n^2+4n+1}=\sum_{n=1}^{+\infty}\arctan\frac{(2n+2)-2n}{1+2n(2n+2)}$$
$$=\sum_{n=1}^{+\infty}\left[\arctan(2n+2)-\arctan(2n)\right]$$

故级数对应的部分和

$$\text{原式}=\lim_{N\to\infty}[\arctan(2N+2)-\arctan 2]=\frac{\pi}{2}-\arctan 2$$

【注】 由$\arctan x+\arctan\dfrac{1}{x}=\dfrac{\pi}{2}(x>0)$，可得$\dfrac{\pi}{2}-\arctan 2=\arctan\dfrac{1}{2}$.

5、微分方程$\begin{cases}(x+1)\dfrac{dy}{dx}+1=2e^{-y}\\ y(0)=0\end{cases}$的解为________.

【参考解答】：**【思路一】**微分方程为可分离变量的微分方程，当$x\neq-1$时分离变量得

$$\frac{dy}{\quad}=\frac{dx}{x+1}，即\frac{e^y dy}{y}=\frac{dx}{x+1}$$

积分得$-\ln\left|2-e^y\right|=\ln\left|C\quad x+1\right|$,整理得$\dfrac{\quad}{2-e^y}=C(x+1)$. 代入$y(0)=0$，得$C=1$，故解为$\dfrac{1}{2-e^y}=x+1$. 当$x=-1$，代入方程得$1=2e^{-y}$，即$y=\ln 2$.

【思路二】当$x=-1$，代入方程得$1=2e^{-y}$，即$y=\ln 2$. 当$x\neq-1$时，改写微分方程表达式，得

$$(x+1)\frac{d(e^y)}{dx}+e^y=2，即\left[(x+1)e^y\right]'=2$$

两边积分可得$(x+1)e^y=2x+C$. 由$y(0)=0$，得$C=1$，从而$e^y=\dfrac{2x+1}{x+1}$，即

$y=\ln\left|\dfrac{2x+1}{x+1}\right|$. 综上得

$$y=\begin{cases}\ln\left|\dfrac{2x+1}{x+1}\right| & x\neq -1\\ \ln 2 & x=-1\end{cases}$$

【思路三】当$x=-1$，代入方程得$1=2\mathrm{e}^{-y}$，即$y=\ln 2$．当$x\neq -1$时，改写微分方程得$\dfrac{\mathrm{d}e^{y}}{\mathrm{d}x}+\dfrac{1}{x+1}e^{y}=\dfrac{2}{x+1}$，令$e^{y}=u$，则

$$\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{1}{x+1}u=\frac{2}{x+1}$$

由一阶线性微分方程通解计算公式，得

$$\begin{aligned}&\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{1}{x+1}u=\frac{2}{x+1}\\ &e^{y}=u=e^{-\int\frac{1}{x+1}\mathrm{d}x}\left(\int\frac{2}{x+1}e^{\int\frac{1}{x+1}\mathrm{d}x}\mathrm{d}x+C\right)\\ &\quad=\frac{1}{x+1}\left(\int 2\,\mathrm{d}x+C\right)=\frac{2x+C}{x+1}\end{aligned}$$

代入$y(0)=0$，得$C=1$，即$e^{y}=\dfrac{2x+1}{x+1}$．综上得

$$y=\begin{cases}\ln\left|\dfrac{2x+1}{x+1}\right| & x\neq -1\\ \ln 2 & x=-1\end{cases}$$

二、(14 分) 设$f(x)=-\dfrac{1}{2}\left(1+\dfrac{1}{\mathrm{e}}\right)+\displaystyle\int_{-1}^{1}|x-t|\mathrm{e}^{-t^2}\mathrm{d}t$，证明：在区间$(-1,1)$内$f(x)$有且仅有两个实根.

【参考证明】：去掉积分中的绝对值，并由积分的线性运算，得

$$\begin{aligned}f(x)&=-\frac{1}{2}\left(1+\frac{1}{e}\right)+\int_{-1}^{x}(x-t)e^{-t^2}\,\mathrm{d}t+{}^{1}(t-x)e^{-t^2}\,\mathrm{d}t\\ &=-\frac{1}{2}\left(1+\frac{1}{e}\right)+x\int_{-1}e^{-t^2}\,\mathrm{d}t-x\int_{x}e^{-t^2}\,\mathrm{d}t-\int_{-1}te^{-t^2}\,\mathrm{d}t+\int_{x}te^{-t^2}\,\mathrm{d}t\\ &=-\frac{1}{2}\left(1+\frac{1}{e}\right)+x\int_{-1}^{0}e^{-t^2}\,\mathrm{d}t+x\int_{0}^{x}e^{-t^2}\,\mathrm{d}t-x\int_{x}^{0}e^{-t^2}\,\mathrm{d}t-x\int_{0}^{1}e^{-t^2}\,\mathrm{d}t\\ &\qquad-\int_{-1}^{0}te^{-t^2}\,\mathrm{d}t-\int_{0}^{x}te^{-t^2}\,\mathrm{d}t+\int_{x}^{0}te^{-t^2}\,\mathrm{d}t+\int_{0}^{1}te^{-t^2}\,\mathrm{d}t\end{aligned}$$

注意到

$$\begin{aligned}&x\int_{-1}^{0}e^{-t^2}\,\mathrm{d}t=x\int_{0}^{1}e^{-t^2}\,\mathrm{d}t,\ x\int_{x}^{0}e^{-t^2}\,\mathrm{d}t=-x\int_{0}^{x}e^{-t^2}\,\mathrm{d}t\\ &\int_{x}^{0}te^{-t^2}\,\mathrm{d}t=-\int_{0}^{x}te^{-t^2}\,\mathrm{d}t,\ \int_{-1}^{0}te^{-t^2}\,\mathrm{d}t=-\int_{0}^{1}te^{-t^2}\,\mathrm{d}t\end{aligned}$$

整理$f(x)$的表达式，得

$$f(x)=2x\int_0^x e^{-t^2}\,\mathrm{d}t-2\int_0^x te^{-t^2}\,\mathrm{d}t+2\int_0^1 te^{-t^2}\,\mathrm{d}t-\frac{1}{2}\left(1+\frac{1}{e}\right)$$

$$=2x\int_0^x e^{-t^2}\,\mathrm{d}t-\left(1-e^{-x^2}\right)+\left(1-\frac{1}{e}\right)-\frac{1}{2}\left(1+\frac{1}{e}\right)$$

$$=2x\int_0^x e^{-t^2}\,\mathrm{d}t+e^{-x^2}-\frac{3}{2e}-\frac{1}{2}$$

$$f'(x)=2\int_0^x e^{-t^2}\,\mathrm{d}t+2xe^{-x^2}-2xe^{-x^2}=2\int_0^x e^{-t^2}\,\mathrm{d}t$$

当$x>0$，则$f'(x)>0$；当$x<0$，则$f'(x)<0$. 即函数如果有零点，则在$x=0$的左右两侧各有一个，故只需考察$x>0$一侧的零点存在性. 当$x=0$时，

$$f(0)=-\frac{1}{2}\left(1+\frac{1}{\mathrm{e}}\right)+\int_{-1}^1 |\,t\,|\mathrm{e}^{-t^2}\mathrm{d}t=-\frac{1}{2}\left(1+\frac{1}{\mathrm{e}}\right)+2\int_0^1 t\,\mathrm{e}^{-t^2}\mathrm{d}t$$

$$=-\frac{1}{2}\left(1+\frac{1}{\mathrm{e}}\right)+\frac{e-1}{e}=\frac{e-3}{2e}<0$$

$$f(1)=2\int_0^1 e^{-t^2}\,\mathrm{d}t-\frac{1}{2e}-\frac{1}{2}>2\int_0^1 e^{-x}\,\mathrm{d}x-\frac{1}{2e}-\frac{1}{2}$$

$$=2-\frac{2}{e}-\frac{1}{2}+\frac{1}{2e}=\frac{3e-5}{2e}>0$$

故由零点定理知，$f(x)$在$(0,1)$上至少有一个零点，综上可知$f(x)$在$(0,1)$有且只有一个零点. 因此，$f(x)$在$(-1,1)$有且只有两个实根.

三、(14分) 设函数$f(x,y)$在闭区域$D=\left\{(x,y)|\,x^2+y^2\le 1\right\}$上具有二阶连续偏导数，且$\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=x^2+y^2$，求$\displaystyle\lim_{r\to 0^+}\frac{\iint_{x^2+y^2\le r^2}\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y}{(\tan r-\sin r)^2}$.

【参考解答】: 令$x=\rho\cos\theta, y=\rho\sin\theta$，记$D_r: x^2+y^2\le r^2$，则

$$D_r: 0\le\theta\le 2\pi, 0\le\rho\le r$$

并记$I=\displaystyle\iint_{x^2+y^2\le r^2}\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$，则由二重积分极坐标计算方法，得

$$I=\int_0^r \rho\,\mathrm{d}\rho\int_0^{2\pi}\left(\rho\cos\theta\frac{\partial f}{\partial x}+\rho\sin\theta\frac{\partial f}{\partial y}\right)\mathrm{d}\theta$$

又$\rho\cos\theta\,\mathrm{d}\theta=\mathrm{d}y, \rho\sin\theta\,\mathrm{d}\theta=-\mathrm{d}x$，记$L_\rho: x^2+y^2=\rho^2$，$D_\rho: x^2+y^2\le\rho^2$，则由对坐标的曲线积分的直接参数方程计算法和格林公式，得

$$\int_0^{2\pi}\left(\rho\cos\theta\frac{\partial f}{\partial x}+\rho\sin\theta\frac{\partial f}{\partial y}\right)\mathrm{d}\theta=\oint_{L_\rho}\frac{\partial f}{\partial x}\mathrm{d}y-\frac{\partial f}{\partial y}\mathrm{d}x$$

$$=\iint_{D_\rho}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)\mathrm{d}x\,\mathrm{d}y=\iint_{D_\rho}\left(x^2+y^2\right)\mathrm{d}x\,\mathrm{d}y$$

$$=\int_0^{2\pi}\mathrm{d}\theta\int_0^\rho t^2\cdot t\,\mathrm{d}t=2\pi\cdot\frac{\rho^4}{4}$$

代入积分式，得

$$I=\int_0^r \rho\cdot 2\pi\cdot\frac{\rho^4}{4}\,\mathrm{d}\rho=\frac{\pi}{2}\int_0^r \rho^5\,\mathrm{d}\rho=\frac{\pi}{12}r^6$$

代入极限式，由 $\sin r\sim r, 1-\cos r\sim\frac{r^2}{2}\left(r\to 0^+\right)$，得

$$原式=\frac{\pi}{12}\lim_{r\to 0^+}\frac{r^6}{(\tan r-\sin r)^2}=\frac{\pi}{12}\lim_{r\to 0^+}\frac{r^6}{\sin^2 r\,\frac{(1-\cos r)^2}{\cos^2 r}}$$

$$=\frac{\pi}{12}\lim_{r\to 0^+}\frac{r^4}{(1-\cos r)^2}=\frac{\pi}{12}\lim_{r\to 0^+}\frac{r^4}{(\frac{r^2}{2})^2}=\frac{\pi}{12}\cdot 4=\frac{\pi}{3}$$

【注】对于其中积分的计算也可以采用分部积分法. 记

$$D_1=\{(x,y)\mid x^2+y^2\le r^2\},$$

由定积分的分部积分公式可得：

$$\iint\limits_{D_1} x\frac{\partial f}{\partial x}\mathrm{d}x\,\mathrm{d}y=\int_{-r}^{r}\mathrm{d}y\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} x\frac{\partial f}{\partial x}\mathrm{d}x$$

$$=\frac{1}{2}\int_{-r}^{r}\mathrm{d}y\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}\frac{\partial f}{\partial x}\mathrm{d}(x^2+y^2)$$

$$=\frac{1}{2}\int_{-r}^{r}\left[(x^2+y^2)\frac{\partial f}{\partial x}\Big|_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}-\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}(x^2+y^2)\frac{\partial^2 f}{\partial x^2}\mathrm{d}x\right]\mathrm{d}y$$

$$=\frac{1}{2}r^2\int_{-r}^{r}[f_x(\sqrt{r^2-y^2},y)-f_x(-\sqrt{r^2-y^2},y)]\mathrm{d}y$$

$$-\frac{1}{2}\int_{-r}^{r}\mathrm{d}y\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}(x^2+y^2)\frac{\partial^2 f}{\partial x^2}\mathrm{d}x$$

注意到 $f_x(\sqrt{r^2-y^2},y)-f_x(-\sqrt{r^2-y^2},y)=\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}\frac{\partial^2 f}{\partial x^2}\mathrm{d}x$，所以

$$\iint\limits_{D_1} x\frac{\partial f}{\partial x}\mathrm{d}x\,\mathrm{d}y$$

$$=\frac{1}{2}r^2\int_{-r}^{r}\mathrm{d}y\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}\frac{\partial^2 f}{\partial x^2}\mathrm{d}x-\frac{1}{2}\int_{-r}^{r}\mathrm{d}y\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}(x^2+y^2)\frac{\partial^2 f}{\partial x^2}\mathrm{d}x$$

$$=\frac{1}{2}r^2\iint\limits_{D_1}\frac{\partial^2 f}{\partial x^2}\mathrm{d}x\,\mathrm{d}y-\frac{1}{2}\iint\limits_{D_1}(x^2+y^2)\frac{\partial^2 f}{\partial x^2}\mathrm{d}x\,\mathrm{d}y$$

$$=\frac{1}{2}\iint\limits_{D_1}[r^2-(x^2+y^2)]\frac{\partial^2 f}{\partial x^2}\mathrm{d}x\,\mathrm{d}y$$

同理 $\iint\limits_{D_1} y\frac{\partial f}{\partial y}\mathrm{d}x\,\mathrm{d}y=\frac{1}{2}\iint\limits_{D_1}[r^2-(x^2+y^2)]\frac{\partial^2 f}{\partial y^2}\mathrm{d}x\,\mathrm{d}y$，因此，

$$\iint_{D_1}\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y=\frac{1}{2}\iint_{D_1}[r^2-(x^2+y^2)](\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2})\mathrm{d}x\,\mathrm{d}y$$
$$=\frac{1}{2}\iint_{D_1}(x^2+y^2)[r^2-(x^2+y^2)]\mathrm{d}x\,\mathrm{d}y=\frac{\pi}{12}r^6$$

四、(14 分) 若对于 R^3 中半空间 $\left\{(x,y,z)\in R^3|\, x>0\right\}$ 内任意有向光滑封闭曲面 S，都有

$$\iint_S xf'(x)\,\mathrm{d}y\,\mathrm{d}z+y\left(xf(x)-f'(x)\right)\mathrm{d}z\,\mathrm{d}x-xz\left(\sin x+f'(x)\right)\mathrm{d}x\,\mathrm{d}y=0,$$

其中 f 在 $(0,+\infty)$ 上二阶导数连续且 $\lim\limits_{x\to0^+}f(x)=\lim\limits_{x\to0^+}f'(x)=0$，求 $f(x)$.

【参考解答】: 记 $P=xf'(x),Q=y\left(xf(x)-f'(x)\right),R=-xz\left(\sin x+f'(x)\right)$，则

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=xf''(x)-xf'(x)+xf(x)-x\sin x$$

由题设可知 $\dfrac{\partial P}{\partial x}+\dfrac{\partial Q}{\partial y}+\dfrac{\partial R}{\partial z}=0$，即

$$f''(x)-f'(x)+f(x)=\sin x$$

由特征方程计算可得该方程对应的齐次线性方程的通解为

$$Y=e^{\frac{1}{2}x}\left(C_1\cos\frac{\sqrt{3}}{2}x+C_2\sin\frac{\sqrt{3}}{2}x\right)$$

令特解为 $y^*=a\cos x+b\sin x$，代入计算得原方程有特解 $y^*=\cos x$，故原方程的通解为

$$f(x)=e^{\frac{1}{2}x}\left(C_1\cos\frac{\sqrt{3}}{2}x+C_2\sin\frac{\sqrt{3}}{2}x\right)+\cos x$$

由已知 $\lim\limits_{x\to0^+}f(x)=\lim\limits_{x\to0^+}f'(x)=0$ 可得 $C_1=-1,C_2=\dfrac{1}{\sqrt{3}}$，即

$$f(x)=e^{\frac{1}{2}x}\left(-\cos\frac{\sqrt{3}}{2}x+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}x\right)+\cos x.$$

五、(14 分) 设 $f(x)=\int_0^x\left(1-\dfrac{[u]}{u}\right)\mathrm{d}u$，其中 $[x]$ 表示小于等于 x 的最大整数，试讨论 $\int_1^{+\infty}\dfrac{\mathrm{e}^{f(x)}}{x^p}\cos\left(x^2-\dfrac{1}{x^2}\right)\mathrm{d}x$ 的敛散性，其中 $p>0$.

【参考解答】: 当 $x\in[N,N+1)$ 时，

$$
\begin{aligned}
f(x) &= \int_0^1 \mathrm{d}u + \int_1^x \left(1-\frac{[u]}{u}\right)\mathrm{d}\,u \\
&= 1+\sum_{k=1}^{N-1}\int_k^{k+1}\left(1-\frac{k}{u}\right)\mathrm{d}\,u + \int_N^x \left(1-\frac{N}{u}\right)\mathrm{d}\,u = x+\ln(N!)-N\ln x
\end{aligned}
$$

于是$\mathrm{e}^{f(x)} = \dfrac{e^x N!}{x^N}, x\in[N,N+1)$，由斯特林(stirling)公式

$$
n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (n\to\infty)
$$

且$\dfrac{e^N N!}{(N+1)^N} \le \mathrm{e}^{f(x)} \le \dfrac{e^{N+1}N!}{N^N}$，从而$x$与$N$充分大时，有

$$
\frac{e^N N!}{(N+1)^N} \sim \frac{1}{e}\sqrt{2\pi}\sqrt{N} \le \frac{1}{e}\sqrt{2\pi}\sqrt{x}
$$

$$
\frac{e^{N+1}N!}{N^N} \sim \sqrt{2\pi}e\sqrt{N} \le \sqrt{2\pi}e\sqrt{x}
$$

从而可知$\mathrm{e}^{f(x)}$与$\sqrt{x}$同阶无穷大，于是$\displaystyle\int_1^{+\infty}\frac{\mathrm{e}^{f(x)}}{x^p}\cos\left(x^2-\frac{1}{x^2}\right)\mathrm{d}x$的敛散性与$\displaystyle\int_1^{+\infty}\frac{1}{x^{p-\frac{1}{2}}}\cos\left(x^2-\frac{1}{x^2}\right)\mathrm{d}x$的敛散性相同. 令$x=\sqrt{y}$，则

$$
\text{原积分} \sim \int_1^{\infty}\frac{\cos\left(y-\frac{1}{y}\right)}{y^{\frac{2p+1}{4}}}\mathrm{d}\,y
$$

由狄利克雷判别法，当$p>0$时，$\displaystyle\int_1^{\infty}\frac{\cos\left(y-\frac{1}{y}\right)}{y^{\frac{2p+1}{4}}}\mathrm{d}\,y$收敛且

$$
\int_1^{\infty}\frac{\cos\left(y-\frac{1}{y}\right)}{y^{\frac{2p+1}{4}}}\mathrm{d}\,y = \int_1^{\infty}\frac{\cos y\cos\frac{1}{y}}{y^{\frac{2p+1}{4}}}\mathrm{d}\,y + \int_1^{\infty}\frac{\sin y\sin\frac{1}{y}}{y^{\frac{2p+1}{4}}}\mathrm{d}\,y
$$

当 $\dfrac{2p+1}{4}>1$ ，即 $p>\dfrac{3}{2}$ 知以上两项均绝对收敛， 对于 $0<p\le\dfrac{3}{2}$ ，$\displaystyle\int_1^{\infty}\frac{\sin y\sin\frac{1}{y}}{y^{\frac{2p+1}{4}}}\mathrm{d}\,y \sim \int_1^{\infty}\frac{\sin y}{y^{\frac{2p+1}{4}+1}}\mathrm{d}\,y$，显然绝对收敛. 但

$$
\int_1^{\infty}\frac{\cos y\cos\frac{1}{y}}{y^{\frac{2p+1}{4}}}\mathrm{d}\,y \sim \int_1^{\infty}\frac{\cos y}{y^{\frac{2p+1}{4}}}\mathrm{d}\,y = \infty
$$

发散，故原无穷积分在$0 < p \le \dfrac{3}{2}$条件收敛，$p > \dfrac{3}{2}$绝对收敛.

六、(14 分) 设正数列$\{a_n\}$单调减少且趋于零，$f(x)=\sum\limits_{n=1}^{\infty} a_n^n x^n$，证明：若级数$\sum\limits_{n=1}^{\infty} a_n$发散，则积分$\int_1^{+\infty} \dfrac{\ln f(x)}{x^2}\mathrm{d}x$也发散.

【参考解答】：级数$\sum\limits_{n=1}^{\infty} a_n^n x^n$的收敛半径$R=\lim\limits_{n\to\infty}\dfrac{1}{\sqrt[n]{a_n^n}}=\lim\limits_{n\to\infty}\dfrac{1}{a_n}=\infty$，所以$f(x)$的定义域是$(-\infty,+\infty)$. 若$x\in\left[\dfrac{e}{a_p},\dfrac{e}{a_{p+1}}\right]$，因$a_n$单调减少，则当$k\le p$时$a_k x \ge a_p x \ge e$. 因此

$$f(x)\ge \sum_{k=0}^{p}\left(a_k x\right)^k \ge \sum_{k=0}^{p} e^k \ge e^p$$

于是$\ln f(x) > p\left(\dfrac{e}{a_p}\le x \le \dfrac{e}{a_{p+1}}\right)$. 又因为当$x\ge 0$时，$f(x)\ge f(0)=1$，所以对固定的$n$，当$X > \dfrac{e}{a_n}$时，

$$\int_1^X \frac{\ln f(x)}{x^2}\mathrm{d}x=\int_1^{\frac{e}{a}}\frac{\ln f(x)}{x^2}\mathrm{d}x+\sum_{p=1}^{n-1}\int_{\frac{e}{a_p}}^{\frac{e}{a_{p+1}}}\frac{\ln f(x)}{x^2}\mathrm{d}x+\int_{\frac{e}{a_n}}^{X}\frac{\ln f(x)}{x^2}\mathrm{d}x$$

$$\ge \sum_{p=1}^{n-1} p\int_{\frac{e}{a_p}}^{\frac{e}{a_{p+1}}}\frac{\mathrm{d}x}{x^2}+n\int_{\frac{e}{a_n}}^{X}\frac{\mathrm{d}x}{x^2}$$

$$=\sum_{p=1}^{n-1} p\left(\frac{a_p}{e}-\frac{a_{p+1}}{e}\right)+n\left(\frac{a_n}{e}-\frac{1}{X}\right)=\frac{1}{e}\sum_{p=1}^{n} a_p-\frac{n}{X}$$

于是当$X > \max\left\{n,\dfrac{e}{a_n}\right\}$时，$\int_1^X \dfrac{\ln f(x)}{x^2}\mathrm{d}x \ge \dfrac{1}{e}\sum\limits_{p=1}^{n} a_p - 1$. 因为级数$\sum\limits_{n=1}^{\infty} a_n$发散，所以$\lim\limits_{X\to\infty}\int_1^X \dfrac{\ln f(x)}{x^2}\mathrm{d}x=\infty$，即积分$\int_1^{+\infty}\dfrac{\ln f(x)}{x^2}\mathrm{d}x$发散.